THE OPERATOR IDEAL $(\Pi_2)^2$ AND KERNELS RELATED TO BESOV FUNCTION SPACES

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Abstract

Let $\Pi_2$ be the operator ideal of absolutely 2-summing operators and $(\Pi_2)^2$ be his square. In this paper, we give sufficient conditions under which certain kinds of integral operator, induced by kernels related to Besov spaces, belongs to $(\Pi_2)^2$. In this context, we also consider the case of a continuous Lipschitz space-valued kernel.

1. Introduction

The dual of a Banach space $E$ is denoted by $E'$. If $T$ is an operator between Banach spaces, $T'$ is the dual of $T$. Concerning the basic definitions from the theory of operator ideals, the reader is referred to [9]. In the following, $\Pi_2$ denote the operator ideals of absolutely 2-summing operators, and $(\Pi_2)^2$ is the square $\Pi_2 \circ \Pi_2$. We recall two properties of
(\Pi_2^2): (a) this approximative operator ideal admits a spectral trace (see [8, (4.a.6)] and [11, (4.2.30)]); and (b) every \( (\Pi_2^2) \)-operator is nuclear (see [9, (24.6.5)]).

We will mainly consider the approximation numbers \( a_n(T) \), the Gelfand numbers \( c_n(T) \), and the Weyl numbers \( x_n(T) \) of an operator \( T \). We mention that \( a := (a_n(T)), c := (c_n(T)), \) and \( x := (x_n(T)) \) are additive and multiplicative \( s \)-number function. If \( s \in \{a, c, x\} \), then we can define the operator ideals \( L_{p,q}^{(s)} \) with \( 0 < p, q \leq \infty \); see [9, 3] and [11, 2].

In the following, \( \Omega \subset \mathbb{R}^N \) is a bounded minimally smooth domain. Let \( E \) by any Banach space. Let \( 0 < \sigma < \infty \) and \( 1 \leq p, q \leq \infty \). We denote by \( L_p(\Omega) \) the spaces of (equivalence classes) of scalar valued Lebesgue measurable functions \( f \) on \( \Omega \) endowed with the norm

\[
|f|_{L_p(\Omega)} := \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}.
\]

The Besov space \( B_{p,q}^{\sigma}(\Omega, E) \) consists of certain \( E \)-valued functions \( f \) defined on \( \Omega \); see [8, (3.b.3)]. If \( E \) is the scalar field, then we simply write \( B_{p,q}^{\sigma}(\Omega) \).

If \( 1 \leq p \leq \infty \), then the dual exponent \( p' \) is determined by the equation \( \frac{1}{p} + \frac{1}{p'} = 1 \).

The letter \( c \) (without or with index) is reserved to denote positive constants with may or may not depend on some fixed parameters and domains, but not on other quantities like operators or natural numbers.

2. Kernels of Besov Type

We start this section with the following theorem:

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be a minimally smooth bounded domain, \( 1 \leq p \leq \infty \) and \( \sigma > \max(0, N(1/2 - 1/p')) \). Let
and suppose that $\alpha < 2$. If $K \in B^\sigma_{2,2}(\Omega; L_p(\Omega))$, then

$$T_K : f(t) \mapsto \int_\Omega K(s, t)f(t)dt,$$

verifies $T_K \in (\prod_2)^2 (L_{p'}(\Omega), L_{p'}(\Omega))$.

**Proof.** By [8, (3.c.5)] (see also [1, p. 262], the Weyl numbers of the Besov embedding

$$I_\Omega : B^\sigma_{2,2}(\Omega) \to L_{p'}(\Omega),$$

satisfies

$$x_n(I_\Omega) \leq \begin{cases} cn^{-\sigma/N}, & \text{if } 1 \leq p' \leq 2, \\ cn^{-(\sigma/N+1/p'-1/2)}, & \text{if } 2 \leq p' \leq \infty, \end{cases}$$

for each $n \in \mathbb{N}$. Hence,

$$I_\Omega \in L_{a,\infty}^{(x)}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)) \subseteq L_{2,1}^{(x)}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)) \subseteq \prod_2(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)),$$

since $\alpha < 2$ and where the last inclusion follows from [11, (2.7.4)] (see also [8, (2.a.11)]).

We have $K \in B^\sigma_{2,2}(\Omega; L_p(\Omega))$. Using [8, (3.d.4)], we get

$$S_K \in \prod_2(L_p(\Omega)^{\prime}, B^\sigma_{2,2}(\Omega)),$$

where $S_K(h') = h' \circ K$, with $h' \in L_p(\Omega)^{\prime}$.

Finally, we consider the factorization

$$T_K = I_\Omega \circ S_K : L_{p'}(\Omega) \xrightarrow{S_K} B^\sigma_{2,2}(\Omega) \xrightarrow{I_\Omega} L_{p'}(\Omega),$$
and we therefore obtain $T_K \in (\prod_2)^2 (L_{p'}(\Omega), L_{p'}(\Omega))$.

\[ \square \]

We also have the following theorem:

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^N$ be a minimally smooth bounded domain, $1 < p \leq 2$ and $\sigma > \max(0, N(1/2 - 1/p'))$. Let

$$1/\beta := \sigma / N + 1/p' - 1/2,$$

and suppose that $\beta < 2$. If $K \in L_p(\Omega; B^\sigma_{2,2}(\Omega))$, then

$$T_K : f(t) \to \int_\Omega K(s, t)f(t)dt,$$

verifies $T_K \in (\prod_2)^2 (L_p(\Omega), L_p(\Omega))$.

**Proof.** By [8, (3.c.5)] (see also [2] and [3]), we know that the Besov embedding $I_\Omega : B^\sigma_{2,2}(\Omega) \to L_{p'}(\Omega)$ satisfies

$$a_n(I_\Omega) \leq cn^{-\sigma/\min(1/p' - 1/2)},$$

for each $n \in \mathbb{N}$. Thus, in view of [11, (2.3.4)] and [11, (2.2.11)], we have

$$I_\Omega \in \ell^{(a)}_{\beta, \infty}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)) \subseteq \ell^{(a)}_{t,1}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega))$$

$$\subseteq \ell^{(c)}_{t,1}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)) \subseteq \ell^{(c)}(B^\sigma_{2,2}(\Omega), L_{p'}(\Omega)),$$

and where the first inclusion follows from [11, (2.2.7)], with $\beta < t < \infty$. Hence, using [11, (2.4.11)], we say that $I_\Omega$ is compact, and then $a_n(I_\Omega) = a_n(I_\Omega)$ (see [11, (2.3.16)]). Therefore,

$$I_\Omega \in \ell^{(a)}_{\beta, \infty}(L_{p'}(\Omega)', B^\sigma_{2,2}(\Omega)')$$

$$\subseteq \ell^{(a)}_{2,1}(L_{p'}(\Omega)', B^\sigma_{2,2}(\Omega)') \subseteq \prod_2(L_{p'}(\Omega)', B^\sigma_{2,2}(\Omega)').$$
Since \( K \in L_p(\Omega; B_{2,2}^\sigma(\Omega)) \), from [11, (6.2.7)] (see also [8, (1.d.5)]), we obtain \( S_K \in \prod_2(B_{2,2}^\sigma(\Omega)^\prime, L_p(\Omega)) \), since \( 1 < p \leq 2 \). Here \( S_K(h') = h' \circ K \), with \( h' \in B_{2,2}^\sigma(\Omega)^\prime \).

Finally, we consider the factorization

\[
T_K = S_K \circ I_\Omega : L_p(\Omega) \xrightarrow{I_\Omega} B_{2,2}^\sigma(\Omega)^\prime \xrightarrow{S_K} L_p(\Omega),
\]

and thus \( T_K \in (\prod_2)^2(L_p(\Omega), L_p(\Omega)) \).

We conclude this section with the following theorem:

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^n \) be a minimally smooth bounded domain, \( \rho, \tau > 0, 1 \leq u, v \leq \infty \), and \( \sigma + \tau > N(1/u - 1/2) \). Let

\[
1/\gamma := \begin{cases} 
\frac{(\sigma + \tau)}{N}, & \text{if } 1 \leq u' \leq 2, \\
\frac{(\sigma + \tau)}{N + 1/u' - 1/2}, & \text{if } 2 \leq u' \leq \infty,
\end{cases}
\]

and suppose that \( \gamma < 2 \). If \( K \in B_{2,2}^\sigma(\Omega; B_{u,v}^\sigma(\Omega)) \), then

\[
T_K : f(t) \mapsto \int_{\Omega} K(s, t)f(t)dt,
\]

verifies \( T_K \in (\prod_2)^2(B_{2,2}^\sigma(\Omega), B_{2,2}^\sigma(\Omega)) \).

**Proof.** By [8, p. 201 and 202], for the embedding

\[
I_B : B_{2,2}^\sigma(\Omega) \to B_{u,v}^\sigma(\Omega)^\prime,
\]

one finds the estimates

\[
x_n(I_B) \leq \begin{cases} 
\left\lfloor cn^{-((\sigma+\tau)/N)} \right\rfloor, & \text{if } 1 \leq u' \leq 2, \\
\left\lfloor cn^{-((\sigma+\tau)/(N+1/u'-1/2))} \right\rfloor, & \text{if } 2 \leq u' \leq \infty,
\end{cases}
\]

for each \( n \in \mathbb{N} \). Hence,
Let $\Omega \subset \mathbb{R}^N$ be a minimally smooth bounded domain. Let $K \in C(\Omega; \mathcal{C}^\lambda(\Omega))$ with $0 < \lambda < 1$. Then the operator
\[
T_K : f(t) \to \int_\Omega K(s, t)f(t)dt,
\]
verifies
\[
T_K \in (\prod_2)^2(L_p(\Omega), L_p(\Omega)),
\]
with 1 < p ≤ 2 and N/p < λ < 1, and

\[ T_K \in (\prod_2)^2(C(\Omega), C(\Omega)). \]

**Proof.** Since \( K \in C(\Omega; C^\lambda(\Omega)) \) implies \( K \in L_p(\Omega, B^\sigma_{2,2}(\Omega)), \) with \( \sigma < \lambda, \) by Theorem 2.2, we obtain \( T_K \in (\prod_2)^2(L_p(\Omega), L_p(\Omega)) \) since 1 < p ≤ 2 and with \( N/p < \sigma. \)

We have \( K \in C(\Omega; C^\lambda(\Omega)). \) Further \( C^\lambda(\Omega) \subseteq B^\sigma_{2,2}(\Omega) \) with \( \lambda > \sigma. \)

Hence \( K \in C(\Omega; B^\sigma_{2,2}(\Omega)) \) and then we can consider \( S_K : B^\sigma_{2,2}(\Omega)' \to C(\Omega), \) where \( S_K \) is the operator \( S_K(h') := h' \circ K \) with \( h' \in B^\sigma_{2,2}(\Omega)' \).

Since \( K \in L_p(\Omega; B^\sigma_{2,2}(\Omega)), \) as in the proof of Theorem 2.2, we also have

\[ T_K = S_K \circ I_\Omega : L_p(\Omega) \xrightarrow{I_\Omega} B^\sigma_{2,2}(\Omega)' \xrightarrow{S_K} C(\Omega), \]

with

\[ I_\Omega \in \prod_2(L_p(\Omega), B^\sigma_{2,2}(\Omega)'). \]

Hence \( T_K \in \prod_2(L_p(\Omega), C(\Omega)). \)

Since 1 < p ≤ 2, the embedding \( i : C(\Omega) \to L_p(\Omega) \) satisfies

\[ i \in \prod_p(C(\Omega), L_p(\Omega)) \subseteq \prod_2(C(\Omega), L_p(\Omega)). \]

Consequently,

\[ T_K = T_K \circ i \in (\prod_2)^2(C(\Omega), C(\Omega)). \]

\[ \square \]

For functions defined on the unit interval \([0, 1]\) and \( p = 2, \) the following result was given, without proof, by Pietsch in [10, p. 88]. This classical result, which (in fact) was obtained by Fredholm [4], is also considered in [6, p. 121] and [7, p. 71].
We have the

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^N$ be a minimally smooth bounded domain. Let $K \in C(\overline{\Omega}; C^\lambda(\overline{\Omega}))$ with $0 < \lambda < 1$.

Then the operator

$$T_K : f(\eta) \mapsto \int_{\overline{\Omega}} K(\eta, \xi)f(\xi)d\xi,$$

verifies $T_K \in (\prod_2)^2(C(\overline{\Omega}), C(\overline{\Omega}))$.

Moreover, if $K_0$ is the restriction of $K$ to $\Omega \times \Omega$, then the operator

$$T_{K_0} : f(\eta) \mapsto \int_{\Omega} K_0(\eta, \xi)f(\xi)d\xi,$$

verifies $T_{K_0} \in (\prod_2)^2(L_p(\Omega), L_p(\Omega))$ with $1 < p \leq 2$ and $N/p < \lambda < 1$.

Further,

$$\sum_{n=1}^\infty \lambda_n(T_K : C(\overline{\Omega}) \to C(\overline{\Omega})) = \int_{\overline{\Omega}} K(\xi, \xi)d\xi.$$

**Proof.** Let $K \in C(\overline{\Omega}; C^\lambda(\overline{\Omega}))$. We consider the product of continuous maps

$$K_0 := R \circ K_{\overline{\Omega}} \circ j : \Omega \overset{i}{\longrightarrow} \overline{\Omega} \overset{K_{\overline{\Omega}}}{\longrightarrow} C^\lambda(\overline{\Omega}) \overset{R}{\longrightarrow} C^\lambda(\Omega),$$

where $j(\xi) := \xi$ for $\xi \in \Omega$, $K_{\overline{\Omega}}(\eta) := K(\eta, \cdot)$ for $\eta \in \overline{\Omega}$, and where $R(f) := f|_\Omega$ (restriction of $f$ to $\Omega$) for $f \in C^\lambda(\overline{\Omega})$.

Since $K_0 \in C(\Omega; C^\lambda(\Omega))$, by Lemma 3.1, we get

$$T_{K_0} \in (\prod_2)^2(L_p(\Omega), L_p(\Omega)),$$

and

$$T_{K_0} \in (\prod_2)^2(C(\Omega), C(\Omega)).$$
Now, we define the product operator

\[ R_K := E \circ T_{K_0} \circ R : C(\overline{\Omega}) \xrightarrow{R} C(\Omega) \xrightarrow{T_{K_0}} C(\Omega) \xrightarrow{E} C(\overline{\Omega}), \]

where \( R \) and \( E \) are the operators

\[ R(g) := g|\Omega \text{ (restriction of } g \text{ to } \Omega ) \text{ for } g \in C(\overline{\Omega}), \]

\[ E(h) := \overline{h} \text{ (extension of } h \text{ to } \overline{\Omega} ) \text{ for } h \in C(\Omega). \]

Since \( K \) is uniformly continuous on \( \overline{K} \times \overline{K} \), it follows that \( T_K = R_K \), and then

\[ T_K \in \left( \prod_2 \right)^2(C(\overline{\Omega}), C(\overline{\Omega})). \]

In order to prove the last assertion in the theorem, about the suite \((\lambda_n)\) of the eigenvalues of \( T_K \), we observe that the spectral (hence continuous, see [10, p. 71]) trace on \( \left( \prod_2 \right)^2 \) of \( T_K \) is (for uniqueness, see [11, p. 173]) the restriction of the continuous trace on the operator ideal \( N \) of nuclear operators; then the last equality follows by [11, p. 274].

References